Taylor series

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1 Introduction

These notes are designed to be brief but self-contained, and deal with Taylor series at a fairly elementary level. They are aimed at second-year students who either have not seen this topic before, or who have seen it but need to brush up. They assume familiarity with basic differentiation and integration. The notes begin with a simple statement of the material you absolutely need to know; this is followed by a development of the theory, which you are encouraged (though not required) to study. If any questions arise, I can be contacted either during the scheduled drop-in sessions, by e-mail (`christopher.hooley@st-andrews.ac.uk`), or at my office (temporarily room 310, but reverting to 304 some time during the semester).

2   What you absolutely need to know

To obtain the Taylor expansion of a function $f(x)$ about the point $x = a$:

- Decide how many terms of the series you're going to generate. Call this number n.

- Obtain the derivatives of the function: $f'(x)$, $f''(x)$, ..., $f^{(n-1)}(x)$.

- Evaluate these at the point a, to obtain $f'(a)$, $f''(a)$, ..., $f^{(n-1)}(a)$.

- The resulting Taylor expansion is given by

$$f(a + x) = f(a) + xf'(a) + \frac{1}{2!}x^2 f''(a) + \ldots + \frac{1}{(n-1)!}x^{n-1} f^{(n-1)}(a) + O(x^n), \quad (1)$$

  where the notation $O(x^n)$ means 'terms of order n or higher in the variable x'.

- In the special case $a = 0$, (1) is called a Maclaurin expansion, but this is a mere terminological matter.

3   Introduction to the theory

Have you ever noticed how some functions 'tell' you how to calculate their values while others don't? For example, consider the function

$$f(x) = x^2. \quad (2)$$

You don't need a table of values for this function. If I ask you to give me $f(7)$, you can tell me straight away that the answer is 49. This is clearly true of any function that involves multiplying, adding, or subtracting the variables that it contains; likewise raising any of them to a positive integer power[1]. Such a function is called a **polynomial**. If $f(x)$ is a polynomial, then its **degree** means the highest power of x that it contains; for example, the function

$$f(x) = 6x^2 - 7x + 4 \quad (3)$$

is of degree two. Some of these have special names:

Degree of polynomial	Name
0	constant
1	linear
2	quadratic
3	cubic
4	quartic
5	quintic

On the other hand, a function like

$$f(x) = \sin x \quad (4)$$

does not 'tell you how' to work out its value. If you want to know the sine of 47 degrees, what do you do? Ask a calculator? But how does the calculator work out the answer? Before electronic

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[1]Raising a number to a power is really just a form of multiplication, provided that the power is a positive integer or zero.

calculators, you had to look the value up in voluminous printed tables. How did they know what to put in those tables?

The answer is contained in the theory of **Taylor series**. It turns out that any smooth, differentiable function can be written as a polynomial; then, by starting from a point where the value of the function is known (for example, it is easy to show that $\sin 45° = \frac{1}{\sqrt{2}}$), the polynomial can be used to find the values near that point. The more accurately we construct the polynomial, the better it will work even far away from the starting point. Let's see in detail how that works.

4   Taylor series: representing non-polynomial functions as polynomials

So, we have a function $f(x)$ and we want to write it as a polynomial. Let's call the point where we start from a, and the distance to the point where we really want to evaluate the function x. Thus, what we really want to know is $f(a+x)$. We are going to assume that this has polynomial form in x:

$$f(a + x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \tag{5}$$

$$= \sum_{n=0}^{\infty} c_n x^n. \tag{6}$$

This polynomial doesn't appear very useful, because (a) it has an infinite number of terms, and (b) we don't know what values the constants c_0, c_1, $c_2 \dots$ take.

Let's deal with problem (b) first. Well, c_0 is easy to determine; we just set $x = 0$ in (5) and all the other terms vanish! Hence

$$c_0 = f(a), \tag{7}$$

the value of the function at the starting point. To get the other coefficients, we knock off powers of x from (5) by differentiating both sides. For example, to get c_1 we differentiate (5) once with respect to x to obtain

$$f'(a + x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots. \tag{8}$$

Now we can use the same trick of setting $x = 0$ to leave only c_1 on the right-hand side, and hence

$$c_1 = f'(a), \tag{9}$$

the first derivative of the function evaluated at the starting point. If we differentiate (8) again, we get

$$f''(a + x) = 2c_2 + 6c_3 x + \dots, \tag{10}$$

and setting $x = 0$ we find

$$2c_2 = f''(a) \qquad \rightarrow \qquad c_2 = \frac{1}{2} f''(a). \tag{11}$$

Notice this factor of two. You should be able to convince yourself that, repeating this differentiation trick an arbitrary number of times, the general expression for the constants c_n is

$$c_n = \frac{1}{n!} f^{(n)}(a), \tag{12}$$

where we have adopted the notation $f^{(n)}(x)$ to mean the nth derivative of $f(x)$.

The conclusion: we can represent $f(a+x)$ exactly as a polynomial providing that we already know the function and all its derivatives at the starting point a. The answer is[2]

$$f(a+x) \;=\; f(a) + xf'(a) + \frac{1}{2}x^2 f''(a) + \frac{1}{6}x^3 f'''(a) + \ldots + \frac{1}{n!}x^n f^{(n)}(a) + \ldots \tag{13}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n. \tag{14}$$

5 An example

For example, let us construct the Taylor series for the function $f(x) = \sin x$, starting at the point $a = \frac{\pi}{4}$. We know the value of the function and all its derivatives at point a:

$$f(x) = \sin x \quad \rightarrow \quad f(a) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}; \tag{15}$$

$$f'(x) = \cos x \quad \rightarrow \quad f'(a) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}; \tag{16}$$

$$f''(x) = -\sin x \quad \rightarrow \quad f''(a) = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}; \tag{17}$$

$$f'''(x) = -\cos x \quad \rightarrow \quad f'''(a) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}; \tag{18}$$

and since the fourth derivative of $\sin x$ is just $\sin x$, the sequence then repeats forever[3]. Taking out the common factor of $\frac{1}{\sqrt{2}}$, we therefore have

$$\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right), \tag{19}$$

with the sign changing every other term[4]. This is sometimes referred to as the **Taylor expansion** of $\sin x$ about the point $\frac{\pi}{4}$.

We could, of course, choose a different starting point for the expansion. Let's do the case $a = 0$; this special case of a Taylor series is often called a **Maclaurin series**. In this case, we have

$$f(x) = \sin x \quad \rightarrow \quad f(a) = \sin 0 = 0; \tag{20}$$
$$f'(x) = \cos x \quad \rightarrow \quad f'(a) = \cos 0 = 1; \tag{21}$$
$$f''(x) = -\sin x \quad \rightarrow \quad f''(a) = -\sin 0 = 0; \tag{22}$$
$$f'''(x) = -\cos x \quad \rightarrow \quad f'''(a) = -\cos 0 = -1; \tag{23}$$

and again, since $f''''(x) = f(x)$, this quartet of coefficients repeats forever. Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \tag{24}$$

This is the Maclaurin expansion of $\sin x$ about the point 0.

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[2]This is just equation (5) copied out again, but replacing the coefficients c_n with their values given by (12).
[3]Ah, the beauty of periodic functions!
[4]If you'd like an exercise, try to write this Taylor series in the form of a sum.

6 Truncating the series, and estimating the error

So that has solved the problem of determining the coefficients, but we still have to face the fact that the series usually has an infinite number of terms, making it slightly impractical. However, if $x \ll 1$ (i.e. if the point we want is rather close to the starting point), then the x^2 term should be smaller than the x term, the x^3 term smaller than the x^2 term, and so forth. In that case, the terms will become so small after a while that they won't be making a significant difference to the answer.

For example, let's calculate $\sin 47°$ to eight decimal places. We should therefore Taylor-expand $\sin x$ around the nearest point for which we know its value, which in this case is $45° = \frac{\pi}{4}$. Copying the series from (19) above:

$$\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right), \tag{25}$$

where now we will substitute in $x = 2° = \frac{\pi}{90}$. Hence[5]

$$x = \frac{\pi}{90} = 0.03490659, \tag{26}$$

$$x^2 = \left(\frac{\pi}{90}\right)^2 = 0.00121847, \tag{27}$$

$$x^3 = \left(\frac{\pi}{90}\right)^3 = 0.00004253, \tag{28}$$

$$x^4 = \left(\frac{\pi}{90}\right)^4 = 0.00000148, \tag{29}$$

$$x^5 = \left(\frac{\pi}{90}\right)^5 = 0.00000005, \tag{30}$$

and we see that the terms from x^6 onwards won't make any difference to the first eight decimal places[6]. Hence we can use the **truncated** form of (25):

$$\sin 47° = \sin\left(\frac{\pi}{4} + x\right) \tag{31}$$

$$\approx \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right) \tag{32}$$

$$= 0.73135370. \tag{33}$$

This is described as the Taylor expansion of $\sin x$ about the point $\frac{\pi}{4}$, **up to and including fifth-order terms**. You are encouraged to reproduce this calculation for yourself, and then check it against the answer your calculator gives when you ask it for $\sin 47°$ directly!

It is sometimes useful to make an estimate of the error made by truncating the Taylor series. Provided that $x \ll 1$, a good estimate of the error is just the first term that we didn't include. So in this case, the error estimate ε is given by the sixth-order term that we dropped[7]:

$$\varepsilon \approx \frac{x^6}{6!} = 2.5 \times 10^{-12}. \tag{34}$$

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[5]All these results are reported to eight decimal places.

[6]In fact, because of the factor $\frac{1}{5!}$ in the Taylor expansion, the x^5 term isn't needed either. But it's better to be on the safe side if in doubt.

[7]The relevant term in the Taylor series would in fact have been negative, but we ignore the sign, since errors have to do with magnitudes.
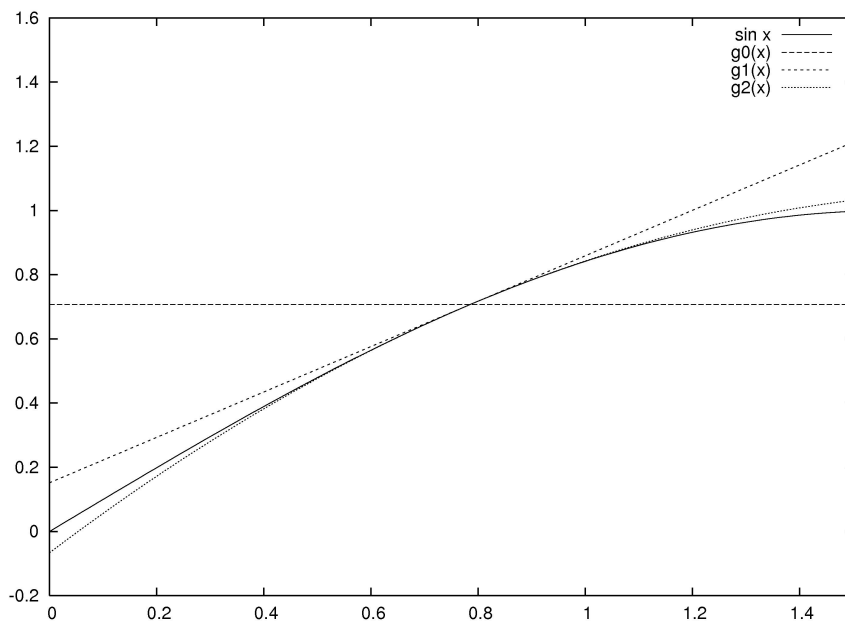
Figure 1: The function $\sin x$, and the first three Taylor approximations to it about the point $\frac{\pi}{4}$.

7 A graphical interpretation

There is a graphical interpretation of Taylor series that some find useful, so let us discuss it briefly. Consider the case that we have been dealing with so far: the expansion of $\sin x$ about the point $\frac{\pi}{4}$. The successive truncations of this Taylor series may be regarded as better and better approximations to the function $\sin x$ about that point:

$$g_0(x) \;=\; \frac{1}{\sqrt{2}}; \tag{35}$$

$$g_1(x) \;=\; \frac{1}{\sqrt{2}}\left(1+x\right); \tag{36}$$

$$g_2(x) \;=\; \frac{1}{\sqrt{2}}\left(1+x-\frac{x^2}{2!}\right); \tag{37}$$

$$\vdots$$

The function itself, and the first three of these approximations, are graphed in Fig. 1. The first approximation, $g_0(x)$, is simply to assume that the function everywhere is equal to its value at the starting point. The second (linear) approximation, $g_1(x)$, adds a slope; the third (quadratic) approximation, $g_2(x)$, adds curvature; and so forth. Eventually, in the limit that all terms in the Taylor expansion are included, the approximate curve falls onto the exact line.

8 Taylor series for a function of a function

A quick note about finding truncated Taylor series for more complicated functions. For example, suppose I want the third-order Maclaurin expansion[8] of the function

$$h(y) = \sin\left[\ln(1+y)\right]. \tag{38}$$

One way to do it would be using the ordinary rules for deriving a Taylor series. The other would be to use the series for $\sin x$ and $\ln(1+y)$, which are easier to derive[9]:

$$\sin x = x - \frac{x^3}{3!} + O(x^5); \tag{39}$$

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + O(y^4). \tag{40}$$

If we substitute the right-hand side of (40) in place of x in (39), we obtain

$$\sin\left[\ln(1+y)\right] = \left(y - \frac{y^2}{2} + \frac{y^3}{3}\right) - \frac{1}{3!}\left(y - \frac{y^2}{2} + \frac{y^3}{3}\right)^3 + O(y^4) \tag{41}$$

$$= y - \frac{y^2}{2} + \frac{y^3}{6} + O(y^4), \tag{42}$$

where to obtain the last line we have multiplied out the brackets and thrown away all terms of higher order than y^3. You are encouraged to derive the Maclaurin series for (38) directly and check that the answer comes out the same.

9 Exercises

9.1 On the need-to-know material

1. Find the Taylor expansion of the function $\cos x$ about the point $x = \frac{\pi}{6}$, to all orders.

2. Find the Taylor expansion of the function $\frac{1}{t}$ about the point $t = 2$, to all orders.

3. Find the Maclaurin expansion of the function $\sqrt{1+x}$, up to and including terms of order x^4.

9.2 More advanced

1. Find the third-order Maclaurin expansion of the function $\frac{1}{1+y}$, and use it to estimate the decimal expansion of $\frac{10}{11}$. Estimate the error in your answer.

2. Using Taylor-series techniques, obtain the value of $\tan 138°$ to five decimal places. (*Hint:* Choose a sensible starting point! A sketch of the angle involved may help.)

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[8]Remember, a Maclaurin expansion is just a Taylor expansion about the origin.

[9]The symbol $O(y^4)$ at the end means that all the omitted terms are of fourth order or higher in y.